III. "On Waves in Liquids." By W. J. Macquorn Rankine, C.E., LL.D., F.R.S. Received April 16, 1868.

(Abstract.)

(1) *Object of this Paper.*—It has long been known that in an uniform canal filled with liquid, the speed of advance of a wave in which the horizontal component of the disturbance is uniform from surface to bottom is equal to the velocity acquired by a heavy body in falling through half the depth of the canal. But, so far as I know, it has not hitherto been pointed out that a similar law exists for waves transmitting a disturbance of any possible kind in a liquid of limited or unlimited depth, provided only that the upper surface of the liquid is a surface of uniform pressure. The object of this paper is to demonstrate that law, and to show some of its applications.

(2) *Velocity of Advance defined.*—Throughout this investigation the velocity of advance of a wave will be defined to be the mean between the velocities with which the shape of the wave advances relatively to a surface-particle at the crest, and to a surface-particle in the trough respectively. In ordinary rolling waves the velocities of particles in those two positions are equal and contrary, so that the speed of advance as above defined is equal to the speed of advance of the wave relatively to the earth. A wave of translation in which the velocities of particles at the crest and hollow are not equal and contrary, may be regarded as produced by compounding the motion of a rolling wave with that of a current whose velocity is half the difference of the velocities of those particles.

(3) *Relation between height of wave and horizontal disturbance at the surface.*—The following relation between the height of a wave and the horizontal disturbance of the surface-particles has already been proved and made use of by various authors; and it is demonstrated here for convenience only. Let $+u_1$ and $-u_1$ be the velocities of a surface-particle at the crest and trough of a wave respectively. Let a be the velocity of advance of the wave as defined in article 2. Conceive a horizontal current with the uniform velocity $-a$ to be combined with the actual wave-motion; the resultant motion is that of an undulating current, presenting stationary waves in its course; and the forces which act on the particles are not altered. The resultant velocity of a particle at the crest becomes $-a+u_1$; and the resultant velocity of a particle in the trough becomes $-a-u_1$. Let the height from trough to crest be denoted by Δz; then, since the upper surface of the liquid is supposed to be a surface of uniform pressure, the principle of the conservation of energy gives the following equation:

$$g\Delta z=\tfrac{1}{2}\{(a+u_1)^2-(a-u_1)^2\}=2au_1. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(4) *Virtual Depth of Uniform Horizontal Disturbance.*—By the phrase "virtual depth of uniform horizontal disturbance," or, for brevity's sake, *virtual depth,* I propose to denote *the depth in the liquid to which an uni-*

form horizontal disturbance would have to extend, in order to make the amount of horizontal disturbance equal to the actual amount. That is to say, conceive that a pair of vertical planes normal to the direction of advance, and each of the breadth unity, coincide at a given instant, one with the trough-line or furrow, and the other with the crest-line or ridge, which bound one of the slopes of a wave. We will suppose this to be the front slope, merely to fix the ideas; for similar reasoning applied to the back slope leads to the same results. At a given depth z below the surface, let $-u''$ be the horizontal velocity with which particles are in the act of passing backwards through the plane of the trough, and $+u'$ the velocity with which particles are passing forwards through the plane at the crest; then the rate by volume at which liquid is passing into the space between those two planes is

$$\int u'dz+\int u''dz;$$

the integrations extending from the surface to the bottom. Let k denote the virtual depth; then

$$k=\frac{\int u'dz+\int u''dx}{2u_1}. \qquad (2)$$

(5) *Relation between Virtual Depth and Speed of Advance.*—In an indefinitely short interval of time dt, the volume of liquid which passes into the space between the two vertical planes mentioned in article 4, is

$$2ku_1dt;$$

and in order to make room for that volume of liquid, the front slope of the wave must sweep in the same interval of time through an equal volume. But the volume swept through by the front of the wave is

$$adt\Delta z;$$

so that, cancelling the common factor dt, we have the following equation:

$$a\Delta z=2ku_1;$$

but, according to equation (1), $\Delta z=\frac{2au_1}{g}$; which value being substituted in the above equation, gives

$$\frac{2a^2u_1}{g}=2ku_1,$$

and therefore

$$\frac{a^2}{g}=k, \text{ and } a=\sqrt{gk}; \qquad (3)$$

so that the *velocity of advance of a wave* (defined as in article 2) *is equal to that acquired by a body in falling through half the virtual depth*; and this is true for all possible waves in which the upper surface is a surface of uniform pressure.

(In article 6 of the paper, the speed of advance of a wave of translation is expressed by combining the speed of a rolling wave, $\sqrt{gk}$, with that of a supposed current, as stated in article 2.

In articles 7, 8, and 9 the law which connects the speed of advance of a wave with the virtual depth is compared with the already known laws of the transmission of rolling waves in water of limited or unlimited depth. The principal results may be summed up as follows. Let T be the periodic time of a wave, in seconds; $h=\frac{gT^2}{4\pi^2}$ the *equivalent pendulum*, that is, the height of the pendulum whose period is the same; $c=\frac{\text{length}}{2\pi}$ the *rolling radius*, being the radius of a circle whose circumference is equal to a wave-length; u_1 the greatest horizontal velocity, and w_1 the greatest vertical velocity of a surface-particle; a the velocity of advance; then

$$a=\sqrt{gk}=\sqrt{\frac{w_1gc}{u_1}}=\frac{w_1}{u_1}\cdot\sqrt{gh}=\frac{w_1}{u_1}\cdot\frac{gT}{2\pi},$$

and

$$k=\frac{w_1}{u_1}\cdot c=\frac{w_1^2}{u_1^2}\cdot h.)$$

(10) *Oblique Advance of Forced Waves.*—Let s be the velocity with which a floating solid body is driven horizontally; the wave which that solid body pushes or drags along with it is forced to advance at the velocity s also; while the virtual depth of disturbance, k, bears some relation to the depth of immersion and figure of the solid body. If the speed of advance corresponding to that depth, $a=\sqrt{gk}$, is less than s, a pair of wave-ridges diverge obliquely from the path of the floating body towards opposite sides; and the sine of the angle which each of those ridges makes with that path is $\frac{a}{s}$. Such is the mode of formation of the obliquely spreading waves which travel along with ships*.

When the velocity of the floating body is less than the speed of advance corresponding to the depth to which it disturbs the liquid in its immediate neighbourhood, it is probable that the virtual depth of disturbance of parts of the liquid beyond the immediate action of the floating body adjusts itself to the velocity, and assumes the value $\frac{s^2}{g}$.

11. *Possibility of Obliquely Advancing Tidal Waves.*—It is possible that instead of a depth less than the virtual depth corresponding to the speed of advance of a tidal wave, the ridge of that wave may place itself in a position oblique to the parallels of latitude, according to the principle stated in article 10. It still remains to be ascertained, by the study of tidal observations, whether such phenomena take place in the tides of the ocean.

12. *Terminal Velocity of Waves.*—It is known that in deep water all waves left free from the action of disturbing forces tend ultimately to assume the condition of free rolling waves whose velocity of advance depends on

* See Watts, Rankine, Napier, and Barnes, 'On Ship-building,' Division I. Article 156, p. 79.

their periodic time, and is expressed by the equation $a=\frac{g\mathrm{T}}{2\pi}$. This, then, may be called the *terminal velocity* of a wave of a given period. It follows that if a wave is raised through the disturbance produced by a solid body, that wave will at first travel with a speed depending on the virtual depth of the original disturbance; but as it advances to a greater and greater distance from the disturbing body, the velocity of advance will gradually approximate to the terminal velocity corresponding to the periodic time, and the virtual depth will continually adjust itself to the changing velocity, and approximate gradually to the equivalent pendulum corresponding to the periodic time. Such is the cause of the forward curvature of the ridges of the obliquely diverging waves which follow a ship*.

May 14, 1868.

Lieut.-General SABINE, President, in the Chair.

The Right Hon. the Earl of Rosse was admitted into the Society.

The following communications were read:—

I. "Scientific Exploration of Central Australia." By Dr. G. Neumayer. Communicated by the President. Received April 20, 1868.

If we look on a map of the Australian continent published ten years ago, we are struck by the immense expanse of land then unexplored; we perceive at a glance that the south-eastern sea-board only of this great continent had then been examined with any degree of accuracy, and that very little was known to us respecting the character of its shores on the west and north-west. In two quarters only had the zeal and daring of the explorer succeeded in forcing a path towards the central portions of this vast territory, Sturt having penetrated as far as 24° South and 138° East, and Gregory as far as 21° South and 128° East. The nature of the country traversed by these two eminent explorers was such as to countenance the supposition, that the interior of Australia was little better than one vast desert, offering almost insurmountable obstacles to exploration. The idea, originally advanced by Oxley, that the greater part of the interior was occupied by vast inland lakes, was then abandoned; and the theory just mentioned took its place. In such a state of utter uncertainty as to the nature of the interior of a vast continent, it is but natural that various theories should be started; and no doubt they will, in the end, help to keep up the spirit for rigorous examination and exploration, yet care must be taken that they do not, by the unfavourable nature of their suppositions, tend to discourage enterprise. From such a danger we had a narrow escape

* This is explained in greater detail in a paper read to the Institution of Naval Architects on the 4th of April 1868.